3.1 – Linear Models

The following initial-value problems apply to various models:

Know these

- $\dfrac{dx}{dt} = kx, \quad x(t_0) = x_0$ growth and decay

- $\dfrac{dT}{dt} = k(T - T_m), \quad T(t_0) = T_0$ Newton's Law of Cooling and Heating

  T_m ← Surrounding temp

- $\dfrac{dA}{dt} =$ rate in − rate out Mixtures

  $A(0) = A_0$

- Electrical circuits

LR-series circuit

$$L\frac{di}{dt} + Ri = E(t)$$

L: inductance (henries h)

R: resistance (ohms Ω)

RC – series circuit

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

C: capacitance (farads f)

t: time

E: voltage (volts v)

i – current = $\dfrac{dq}{dt}$ amps

q – charge – coulombs

4. The population of bacteria in a culture grows at a rate proportional to the number of bacteria present at time t. After 3 hours it is observed that 400 bacteria are present. After 10 hours 2000 bacteria are present. What was the initial number of bacteria?

Find P_0

$$\frac{dP}{dt} = kP \qquad \text{separable} \qquad (\text{\& linear})$$

$$\frac{dP}{P} = k\,dt \Rightarrow \ln|P| = kt + C_1$$

$$P(t) = P_0 e^{kt}$$ initial population

$P(3) = 400$

$P(10) = 2000$

$$2000 = P_0 e^{10k}$$

$$400 = P_0 e^{3k}$$

$$5 = e^{7k} \Rightarrow k = \frac{1}{7}\ln 5$$

$$400 = P_0 e^{3(\frac{1}{7}\ln 5)}$$

$$400 = P_0 e^{\ln 5^{3/7}} \Rightarrow P_0 = \frac{400}{5^{3/7}}$$

$P_0 \approx 200.68 \Rightarrow$ 201 bacteria decay: half-life λ (lambda)

$$P(\lambda) = \tfrac{1}{2} P_0$$

$$\tfrac{1}{2} P_0 = P_0 \, e^{\lambda k} \implies k = \frac{1}{\lambda} \ln \frac{1}{2}$$

23. A large tank is filled to capacity with 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped into the tank at a rate of 5 gal/min. The well-mixed solution is pumped out at the same rate. Find the number $A(t)$ of pounds of salt in the tank at time t.

salt solution

$$\frac{2\,lb}{gal} \cdot \frac{5\,gal}{min} = 10\ lb/min \quad \text{rate in}$$

$$\frac{dA}{dt} = \text{rate in} - \text{rate out}$$

$$\frac{A\,lb}{500\,gal} \cdot \frac{5\,gal}{min} = \frac{A}{100}\ lb/min \quad \text{rate out}$$

$$\frac{dA}{dt} = 10 - \frac{A}{100} \qquad \text{(linear)}$$

$$\frac{dA}{dt} + \frac{1}{100} A = 10$$

$$\frac{d}{dt}\left(e^{1/100\, t} A\right) = 10\, e^{1/100\, t}$$

$$e^{1/100\,t} A = 1000 e^{1/100\,t} + C$$

$$A(t) = 1000 + C e^{-1/100\,t}$$

Pure water: $A(0) = 0 \implies C = -1000$

$$A(t) = 1000 - 1000\, e^{-\frac{1}{100} t}$$

25. Solve Problem 23 under the assumption that the solution is pumped out at a faster rate of 10 gal/min. When is the tank empty?

rate in $= 10$ lb/min

$$\text{rate out} = \frac{A}{500-5t} \cdot \frac{10 \text{ gal}}{\text{min}}$$

losing 5 gal every minute: 5 gal in − 10 gal out = −5

$$\frac{dA}{dt} = 10 - \frac{2A}{100-t}$$

(Linear) $$\frac{dA}{dt} + \frac{2}{100-t} A = 10$$

$$\mu = e^{\int \frac{2}{100-t} dt} = e^{-\ln|100-t|^2}$$

$u = 100 - t \Rightarrow du = -dt$

$$\mu = \frac{1}{(100-t)^2}$$

$$\frac{d}{dt}\left[\frac{1}{(100-t)^2} A\right] = \frac{10}{(100-t)^2}$$

$$\frac{1}{(100-t)^2} A = \frac{10}{100-t} + C$$

$$A = 10(100-t) + c(100-t)^2$$

$A(0) = 0 \Rightarrow$ we find that

$$A(t) = 1000 - 10t - \frac{1}{10}(100-t)^2$$

A 200-volt electromotive force is applied to an *RC*-series circuit in which the resistance is 1000 ohms and the capacitance is 5×10^{-6} farad. Find the charge $q(t)$ on the capacitor if $\boxed{i(0) = 0.4.}$ Determine the charge and current at $t = 0.005$ s. Determine the charge as $t \to \infty$.

q : charge

$i = \frac{dq}{dt}$: current

$C = 5 \times 10^{-6} = \frac{5}{10^6} \Rightarrow \frac{1}{C} = \frac{1}{5} \times 10^6 = \frac{10^6}{5}$

RC - series circuit : $R\frac{dq}{dt} + \frac{1}{C}q = E(t)$

$$1000\frac{dq}{dt} + 2 \times 10^5 q = 200$$

$$\frac{dq}{dt} + 200q = 0.2 \qquad \mu = e^{200t}$$

$$\frac{d}{dt}\left(e^{200t} q\right) = \frac{1}{5} e^{200t}$$

$$e^{200t} q = \frac{1}{1000} e^{200t} + C$$

$$q = \frac{1}{1000} + c\, e^{-200t}$$

$i(0) = 0.4$

$$i = \frac{dq}{dt} = -200\, c\, e^{-200t}$$

$$\frac{2}{5} = -200\, c \implies c = -\frac{1}{500}$$

Charge: $\boxed{q(t) = \frac{1}{1000} - \frac{1}{500} e^{-200t}}$

as $t \to \infty$, $q \to \frac{1}{1000}$ Coulombs

$q(0.005)$

$i(0.005) = q'(0.005)$